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A Fock-space theory of local contextual hidden variables

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Abstract. Within the framework of the Fock space formalism of the non-relativistic quantum field theory the problem of hidden variables is studied. It is shown that one can construct, both for fermionic and bosonic Fock spaces, a local completely causal underlying theory which reproduces all quantum probabilities. This is done by suitable generalization of the probability concepts.

1. Introduction

It is a well known fact that quantum mechanical stochasticity can be interpreted as a consequence of a lack of knowledge about some 'hidden variables' of the physical system [1, 2]. More precisely, it is possible to introduce a hidden variables space Ω with the following properties:

(i) if a measurement arrangement and the point $\omega \in \Omega$ are given, then the measurement result for an individual system is completely determined;

(ii) for each statistical operator ρ in the Hilbert state space, there is a probability measure μ_ρ on Ω such that the mean value $\langle \hat{a} \rangle_\rho = \mathrm{Tr}(\hat{a}\rho)$ of an observable $\hat{a}$ in ρ can be computed in terms of Ω and μ_ρ.

However, each subquantum theory of the kind mentioned must be non-local, because any local theory satisfies the Bell inequalities [3], which quantum mechanics violates.

It is very important to point out that this last statement is true in the framework of Kolmogorovian probability and it is not true in the framework of some more general probability concepts; according to Pitowsky [4] and Gudder [5, 6], generalization of probability allows Bell inequalities to be overcome in some special cases.

This paper is closely related to this line of thinking. Its aim is to show that the Bell inequalities argument can be completely overcome by the use of certain generalizations of probability theory. In other words, there is still a possibility of the existence of a consistent local causal subquantum theory that is in agreement with quantum mechanics. We show this explicitly, by constructing an example of such a theory based on a 'contextual' modification of classical statistics.

Let us outline the contents of this paper.

In section 2 we introduce the quantum structure with which we are dealing, in particular the quantum representatives of the measurement arrangements. For a description of the quantum level, we choose the Fock-space formalism.

In section 3 we introduce, motivated by the construction of Gudder [2], the space Ω. It is natural to interpret the points of this as possible subquantum states of the quantum field. As we shall see, this interpretation is compatible with the requirements of (subquantum) locality.

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In section 4 we analyse the possibility of the lack-of-knowledge interpretation of the quantum states. We show that, within the framework of the mentioned modification of statistics, all quantum states can be interpreted in such a manner.

2. The quantum structure

The main subject of our discussion is the quantum field whose quanta are objects existing in the Euclidean three-space E^3. Their internal degrees of freedom are represented by some finite-dimensional Hilbert space W.

The Hilbert space of all one-particle states is $X = L^2(E^3, W)$, that is, the space of all square-integrable functions $f: E^3 \to W$ with a scalar product given by $\langle f, g\rangle = \int (f(x), g(x))$, where $(,)$ denotes the scalar product in W. For the Hilbert space H of all quantum states of the field, we take the Fock (bosonic or fermionic) Hilbert space associated with X.

For each open set $U \subseteq E^3$, let X_U be the Hilbert space of one-particle states which are completely localized in U (actually, $X_U = L^2(U, W)$) and H_U the corresponding Fock space. The space H_U is a subspace of H. Also, if U and V are disjoint open subsets of E^3, then there is a natural isomorphism $F_{U,V}: H_{U\cup V} \to H_U \otimes H_V$. More generally, each orthogonal decomposition of the one-particle Hilbert space induces canonically a tensor decomposition of the whole Fock space (see the appendix).

We shall assume that simple quantum measurements are performed in open connected and bounded regions in E^3. Let τ be the family of all regions of this kind. For reasons of simplicity, we shall confine ourselves to complete local observables with purely discrete spectra. More general cases can be treated similarly. In other words, for the representatives of quantum measurements performed in some region $U \in \tau$, we take the decompositions of H_U into orthogonal sums of one-dimensional subspaces. Equivalently, we can say that simple quantum measurements performed in U are represented by maximal atomic Boolean sub-σ-algebras of the projector lattice $P(H_U)$.

Of course, we have to consider the coincidence measurements as well. We shall suppose that each coincidence measurement is composed from simple measurements performed in regions $U_1, \ldots, U_k \in \tau$ with mutually disjoint closures. If local measurements are coincidentally performed in such regions $U_1, \ldots, U_k \in \tau$ and are represented by maximal atomic Boolean sub-σ-algebras $B_1 \subseteq P(H_{U_1}), \ldots, B_k \subseteq P(H_{U_k})$, then for the representative of the coincidence measurement we can take the set $\{B_1, \ldots, B_k\}$.

Let S be the collection of all such sets $\{B_1, \ldots, B_k\}$. The elements of S we shall call *contexts*. Representatives of simple quantum measurements, that is elements of the form $\{B\}$, we shall call *local contexts*. The contexts are in one-one correspondence with the decomposable maximal atomic Boolean subalgebras of $P(H_{U_1\cup\ldots\cup U_k}) = P(H_{U_1}\otimes\ldots\otimes H_{U_k})$: $\{B_1, \ldots, B_k\} \leftrightarrow B_1\otimes\ldots\otimes B_k$. Also, we can naturally embed these structures into the whole projector lattice $P(H)$ with the help of the decomposition $H = H_{U_1}\otimes\ldots\otimes H_{U_k}\otimes H'$ associated with the orthogonal decomposition $X = X_{U_1}\oplus \ldots\oplus X_{U_k}\oplus X'$.

3. The subquantum space

Now, we construct the subquantum space. For a given local context B, the set of all atoms of B is denoted by Z_B. Let $\Omega = \Pi_B Z_B$, where the product is taken over all local contexts. Consider the family $\boldsymbol{P}$ of all subsets Λ of Ω of the form $\Lambda = \pi^{-1}_{B_1,\dots,B_k}(\Lambda')$, where $\{B_1, \dots, B_k\} \in S$, $\Lambda' \subseteq Z_{B_1} \times \dots \times Z_{B_k}$ and $\pi_{B_1,\dots,B_k} : \Omega \to Z_{B_1} \times \dots \times Z_{B_k}$ is the natural projection. For each fixed $\bar{B} = \{B_1, \dots, B_k\} \in \boldsymbol{S}$ let $\boldsymbol{P}_{\bar{B}}$ be the following subfamily of $\boldsymbol{P}$:

$$\boldsymbol{P}_{\bar{B}} = \{\Lambda \subseteq \Omega \mid \Lambda = \pi^{-1}_{B_1,\dots,B_k}(\Lambda'),\ \Lambda' \subseteq Z_{B_1} \times \dots \times Z_{B_k}\}.$$

The most important properties of the pair $(\Omega, \{\boldsymbol{P}_{\bar{B}};\ \bar{B} \in \boldsymbol{S}\})$ are:

(i) for each $\bar{B} \in S$, the family $\boldsymbol{P}_{\bar{B}}$ is a Boolean σ-algebra of subsets of Ω and $\emptyset, \Omega \in \boldsymbol{P}_{\bar{B}}$;

(ii) if $\bar{B}_1 \subseteq \bar{B}_2$, then $\boldsymbol{P}_{\bar{B}_1} \subseteq \boldsymbol{P}_{\bar{B}_2}$.

As we mentioned before, each context $\bar{B} = \{B_1, \dots, B_k\} \in \boldsymbol{S}$ can be naturally viewed as a Boolean sub-σ-algebra of the projector lattice $P(H)$. In terms of this identification, subsets of $Z_{B_1} \times \dots \times Z_{B_k}$ are in one-one correspondence with projectors in $\bar{B}$. In particular, points of $Z_{B_1} \times \dots \times Z_{B_k}$ correspond to the atoms of $\bar{B}$. For each $S \subseteq Z_{B_1} \times \dots \times Z_{B_k}$, we denote by $c_{\bar{B}}(S)$ the corresponding element of $\bar{B}$.

The just constructed object $(\Omega, \{\boldsymbol{P}_{\bar{B}};\ \bar{B} \in \boldsymbol{S}\})$ plays the role of the subquantum space in our theory. We interpret the points of Ω as possible *individual* quantum field configurations. It is then natural, in the sense of the construction $(\Omega, \{\boldsymbol{P}_{\bar{B}};\ \bar{B} \in \boldsymbol{S}\})$, to formulate the following actualization of the quantum properties.

Postulate. If the context $\bar{B} = \{B_1, \dots, B_k\}$ is chosen and if the quantum field is in the subquantum state $\omega \in \Omega$, then the elementary local quantum properties of ω in the context $\bar{B}$ are respectively $\pi_{B_1}(\omega), \dots, \pi_{B_k}(\omega)$.

The postulate preserves locality in the sense that actualization of a local quantum property in a given subquantum state depends on the corresponding local context only.

According to this postulate, the sets belonging to the family $\boldsymbol{P}$ are those and only those which can be defined by one measurement context only. Naturally, this is the domain of the quantum description. Consequently, each set $\Lambda \subseteq \boldsymbol{P}$ can be interpreted as a projector $c(\Lambda)$ in H. More precisely, for $\Lambda \in \boldsymbol{P}$ we consider a context $\bar{B} = \{B_1, \dots, B_k\} \in \boldsymbol{S}$ with the property $\Lambda \in \boldsymbol{P}_{\bar{B}}$. Then we define $c(\Lambda) = c_{\bar{B}}[\pi_{B_1,\dots,B_k}(\Lambda)]$.

A consequence of this definition is that for each $\bar{B} \in S$, the 'quantum interpretator' map $c: \boldsymbol{P} \to P(H)$ is an isomorphism of $\boldsymbol{P}_{\bar{B}}$ and $\bar{B}$ (viewed here as a Boolean sub-σ-algebra of the projector lattice $P(H)$).

4. The lack of knowledge interpretation of the quantum probabilities

In the background of all hidden variable approaches to quantum theory, there is the lack of knowledge interpretation of the quantum probabilities. The simplest and the best known way of describing lack of knowledge is the classical (Kolmogorovian) probability theory. To be concrete, we consider the triplet $(X, \boldsymbol{T}, \mu)$ where X is a non-empty set, $\boldsymbol{T}$ is a Boolean σ-algebra of subsets of X such that $\emptyset, X \in \boldsymbol{T}$ and $\mu : \boldsymbol{T} \to [0, 1]$ is a positive normalized ($\mu(X) = 1$) σ-additive function (probability measure). We then interpret the points of $\boldsymbol{X}$ as *elementary events* and the sets belonging to $\boldsymbol{T}$ as *events that can be actualized in the considered situation.* In each single experiment each event $\Lambda \in \boldsymbol{T}$ may be actualized or not, and $\mu(\Lambda)$ is the probability of its actualization.

Let us now discuss how this concept can be justified in the present subquantum approach.

First of all, an important point is that we have to consider only 'the quantum stochasticity' that is, *there are no 'non-quantum' events that have probabilities which we should interpret.* In each quantum experiment only the *quantum* events can be actualized. From this point of view, it is natural to take the family $\boldsymbol{P}$ (which is not a Boolean σ-algebra) to be the domain of all probability measures.

Secondly, the events that can be actualized in a given context $\bar{B} \in \boldsymbol{S}$ are those belonging to the Boolean σ-algebra $\boldsymbol{P}_{\bar{B}}$. When $\bar{B} \in \boldsymbol{S}$ is specified, the classical statistics *must be* used. We think that this is sufficient motivation for the following definition.

Definition. A probability measure on $(\Omega, \{\boldsymbol{P}_B\,;\, B \in \boldsymbol{S}\})$ is a map $\mu : \boldsymbol{P} \to [0, 1]$ with the property that its restriction to each $\boldsymbol{P}_B$ is an ordinary probability measure.

It is easy to see that each point $\omega \in \Omega$ canonically determines the probability measure δ_ω on $(\Omega, \{\boldsymbol{P}_B\,;\, B \in \boldsymbol{S}\})$: $\delta_\omega(\Lambda) = \kappa_\Lambda(\omega)$, where κ_Λ is the characteristic function of $\Lambda \in \boldsymbol{P}$. The probability measure δ_ω is concentrated at the point ω and corresponds to the situations in which ω is completely known.

On the other hand, *each statistical operator ρ in H gives rise to a probability measure μ_ρ on* $(\Omega, \{\boldsymbol{P}_B\,;\, B \in \boldsymbol{S}\})$: $\mu_\rho(\Lambda) = \mathrm{Tr}(\rho c(\Lambda))$.

Confirming that the notion of the probability measure on $(\Omega, \{\boldsymbol{P}_B\,;\, B \in \boldsymbol{S}\})$ is the natural framework for describing lack of knowledge about points $\omega \in \Omega$, we can summarize our discussion in the following theorem.

Theorem. The triplet $((\Omega, \{\boldsymbol{P}_B\,;\, B \in \boldsymbol{S}\}), \boldsymbol{P}, c)$ is a local hidden variable theory for the quantum structure described by $(E^3, X, H, \{H_U\,;\, U \in \tau\})$.

5. Concluding remarks

The most important result of this article is the theorem establishing that the modification of the probability concepts enables one to avoid, at the formal level, arguments of the type of the Bell inequalities [3] against the expounded local causal hidden variables theory.

A very important property of the constructed subquantum theory is that the quantum events (projectors in H) can be interpreted in terms of Ω only if, in addition, the corresponding measurement context is specified. In other words, for a given $\omega \in \Omega$ we may have two contexts B_1 and B_2 such that a quantum event $\hat{P}$ occurs in ω, within the context B_1, and does not occur in ω within the context B_2. This is unavoidable, because a hidden variable theory without this property is not possible according to the proofs of Gleason [7] and Bell [8], and of Kochen and Specker [9]. But, as we mentioned before, if we measure some local observable, for example a projector $\hat{P} \in H_U$ in some region $U \in \tau$, on the system in the subquantum state $\omega \in \Omega$, the value of $\hat{P}$ in ω depends on the local measurement context in U only. Therefore, we are dealing with *local contextuality.*

The theory presented admits a relativistically covariant formulation. A solution of this problem is presented in [10], within the framework of algebraic quantum field theory.

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Appendix

For the reader's convenience, a proof of the decomposition property of Fock spaces that we used in the text is given.

Let X be a separable Hilbert space and H the bosonic or the fermionic Fock space associated with X: $H = \Sigma_{k=0}^{\infty} S(X^k)$ (in the bosonic case) and $H = \Sigma_{k=0}^{\infty} A(X^k)$ (in the fermionic case), where $S(X^k)$ denotes the subspace of all symmetric vectors of the kth tensor power X^k of X and $A(X^k)$ denotes the subspace of all antisymmetric vectors of X^k.

With each $\psi \in X$ we can canonically associate an annihilation and (its adjoint) creation operator $\hat{a}(\psi)$ and $\hat{a}^+(\psi)$ in H. These operators satisfy the following relations:

$$
\begin{aligned}
&\hat{a}(\psi)|0\rangle = 0 \qquad \text{for each } \psi \in X, \text{ where } |0\rangle \text{ is the vacuum state in } H \\
&\hat{a}(c_1\psi_1 + c_2\psi_2) = c_1^* \hat{a}(\psi_1) + c_2^* \hat{a}(\psi_2) \\
&[\hat{a}(\psi_1), \hat{a}^+(\psi_2)]_{\mp} = \langle \psi_1 | \psi_2 \rangle \hat{I} \\
&[\hat{a}(\psi_1), \hat{a}(\psi_2)]_{\mp} = 0 \qquad \text{for each } c_1, c_2 \in C \text{ and } \psi_1, \psi_2 \in X.
\end{aligned} \tag{A1}
$$

Now, let us suppose that the space X is decomposed into a direct sum of two orthogonal subspaces X_1 and X_2. Let H_1 and H_2 be the Fock spaces associated with X_1 and X_2 and $\psi_1 \to \hat{a}_1(\psi_1)$ and $\psi_2 \to \hat{a}_2(\psi_2)$ the corresponding annihilation operators in H_1 and H_2. For each $\psi \in X$ we define the operator $\hat{b}(\psi)$ in $H_1 \otimes H_2$:

$$\hat{b}(\psi) = \hat{a}_1(\psi_1) \otimes I_2 + \alpha \otimes \hat{a}_2(\psi_2).$$

Here, $\psi = \psi_1 + \psi_2$, $\psi_{1,2} \in X_{1,2}$, $\alpha = \hat{I}_1$ in the bosonic case and $\alpha = \exp(i\pi \hat{N}_1)$ in the fermionic case, where $\hat{N}_1$ is the corresponding particle number operator. One can easily show that the operators $\hat{b}(\psi)$ and $\hat{b}^+(\psi)$ satisfy the relations (A1) with $|0\rangle_1 \otimes |0\rangle_2$ as a vaccum state. With the help of these relations, it can be shown that the relation:

$$\hat{b}^+(\psi_1) \ldots \hat{b}^+(\psi_k) |0\rangle_1 \otimes |0\rangle_2 \overset{\hat{F}}{\to} \hat{a}^+(\psi_1) \ldots \hat{a}^+(\psi_k) |0\rangle$$

consistently and uniquely defines an isomorphism $\hat{F}: H_1 \otimes H_2 \to H$ with the property $\hat{F}|0\rangle_1 \otimes |0\rangle_2 = |0\rangle$ and $\hat{a}(\psi) = \hat{F}\hat{b}(\psi)\hat{F}^{-1}$, for each $\psi \in X$.

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